MATHEMATICS

REMARKS ON SETS CONVEX IN THE SENSE OF J. DE GROOT

BY

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Let M be a metric space with distance function ϱ. A subset $S \subset M$ is called convex (relative to ϱ) if for every $x, y \in S$ and every $z \in M \setminus S$ the following inequality holds:

$$\varrho(x, y) < \varrho(x, z) + \varrho(z, y).$$

This definition is due to J. DE GROOT [1].

The aim of the present note is to give a positive solution of the following problem raised by J. DE GROOT:

Given a metrizable space M and a closed subset $S \subset M$. Can every (topology-preserving) metric ϱ_s on S be extended to a (topology-preserving) metric on M, under which S is convex?

The proof follows simply from a theorem of HAUSDORFF [2]. In fact, using a well-known theorem of Hausdorff we can extend the metric ϱ_s to a metric ϱ on the whole space.

Let

$$\varphi(x, y) = \min\ (\varrho(x, y),\ \delta(x) + \delta(y)),$$

where $\delta(p) = \varrho(p, S)$ i.e. $\delta(p) = \inf\limits_{t \in S} \varrho(p, t),\ p \in M$.

As has been proved by HAUSDORFF ([2], p. 354) the function φ is a pseudometric, i.e. it is non-negative, symmetric and it satisfies the triangle inequality. Moreover, $\varphi(x, y) = 0$ if and only if $x, y \in S$.

Put for each $x, y \in M$

$$\varrho^*(x, y) = \varrho(x, y) + \varphi(x, y).$$

It is easy to verify that ϱ^* is a topology-preserving metric on M, since $\varrho(x, y) \leqslant \varrho^*(x, y) \leqslant 2\varrho(x, y)$, and ϱ^* is an extension of ϱ_s. Moreover, for every $x, y \in S$ and $z \in M \setminus S$ we obtain the inequality

$$\begin{aligned} \varrho^*(x, y) = \varrho_s(x, y) = \varrho(x, y) \leqslant \varrho(x, z) + \varrho(z, y) < \\ < \varrho(x, z) + \varphi(x, z) + \varrho(z, y) + \varphi(z, y) = \\ = \varrho^*(x, z) + \varrho^*(z, y). \end{aligned}$$

Consequently, the set S is convex relative to ϱ^*, q.e.d.

Remark 1. Defining for $x, y \in M$

$$\hat{\varrho}(x, y) = \varrho(x, y) + \sqrt{\varphi(x, y)}$$

we see that $\hat{\varrho}$ is an extension of ϱ_S, while, moreover, the set S is convex relative to $\hat{\varrho}$ and also every subset $U \subset M \setminus S$ is convex relative to $\hat{\varrho}$. The metrix $\hat{\varrho}$ is suggested by E. Marczewski.

Remark 2. As I have been informed by J. de Groot, the following stronger formulation of the original problem, given by E. Marczewski is true:

Given a metrizable space M and a closed subset $S \subset M$. Can every (topology-preserving) metric ϱ_S on S be extended to a (topology-preserving) metric $\tilde{\varrho}$ on M in such a manner that

$$\tilde{\varrho}(x, y) = \tilde{\varrho}(x, z) + \tilde{\varrho}(z, y)$$

implies $x, y, z \in S$?

We are going to prove that the non-negative, symmetric function

$$\tilde{\varrho}(x, y) = \sqrt{\varrho^2(x, y) + \varphi(x, y)}$$

given by J. de Groot, satisfies the stronger thesis of E. Marczewski (A similar proof was given by A. B. de Miranda.)

Since $\varphi(x, y) = 0$ for every $x, y \in S$,

$$\tilde{\varrho}(x, y) = \varrho(x, y) = \varrho_S(x, y)$$

on S.

Let x, y, z be any points not belonging simultaneously to S. We shall show that

$$[\tilde{\varrho}(x, y)]^2 < [\tilde{\varrho}(x, z) + \varrho(z, y)]^2.$$

Namely:

$$[\tilde{\varrho}(x, y)]^2 = [\sqrt{\varrho^2(x, y) + \varphi(x, y)}]^2 = \varrho^2(x, y) + \varphi(x, y) \leqslant$$

$$\leqslant [\varrho(x, z) + \varrho(z, y)]^2 + \varphi(x, z) + \varphi(z, y) =$$

$$= \varrho^2(x, z) + 2\varrho(x, z) \cdot \varrho(z, y) + \varrho^2(z, y) + \varphi(x, z) + \varphi(z, y) <$$

$$< \varrho^2(x,z) + \varphi(x,z) + 2\sqrt{\varrho^2(x,z) + \varphi(x,z)} \cdot \sqrt{\varrho^2(z, y) + \varphi(z, y)} + \varrho^2(z, y) + \varphi(z, y) =$$

$$= [\sqrt{\varrho^2(x, z) + \varphi(x, z)} + \sqrt{\varrho^2(z, y) + \varphi(z, y)}]^2 = [\tilde{\varrho}(x, z) + \tilde{\varrho}(z, y)]^2.$$

The sign $<$ in the preceding inequalities follows from the fact that for every three points x, y, z not belonging simultaneously to S either $\varphi(x, z) > 0$ or $\varphi(z, y) > 0$.

Remark 3. As we have observed with A. Lelek, we obtain using the function φ an obvious proof of the following identification theorem known only for separable metric spaces; [3] p. 138–139:

Given a metric space M with metric ϱ and a closed set $S \subset M$. There exist a metric space M' with metric ϱ' and a continuous map f of M on M' such that $f(S) = (p_0)$ and f is a homeomorphism on $M \setminus S$.

It is easy to see that for

$$M' = (M \setminus S) \cup (p_0),$$

$$f(x) = \begin{cases} x & \text{for } x \in M \setminus S \\ p_0 & \text{for } x \in S, \end{cases}$$

$$\varrho'(x, y) = \varphi(f^{-1}(x), f^{-1}(y)),$$

the theorem is true.

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REFERENCES

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3. Kuratowski, C., Topologie I, Warszawa-Wrocław 1948.